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Stability and bifurcation in a delayed predator–prey system with Beddington–DeAngelis functional response ☆

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Abstract

We consider a delayed predator–prey system with Beddington–DeAngelis functional response. The stability of the interior equilibrium will be studied by analyzing the associated characteristic transcendental equation. By choosing the delay τ as a bifurcation parameter, we show that Hopf bifurcation can occur as the delay τ crosses some critical values. The direction and stability of the Hopf bifurcation are investigated by following the procedure of deriving normal form given by Faria and Magalhães. An example is given and numerical simulations are performed to illustrate the obtained results.

Keywords: Predator–prey system; Time delay; Stability; Hopf bifurcation; Normal form

1. Introduction

The main purpose of this paper is to study the stability and bifurcation of the following delayed predator–prey system with Beddington–DeAngelis functional response:

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$$\begin{cases} x'(t) = x(t)(1 - x(t)) - \frac{sx(t)y(t)}{x(t)+By(t)+A}, \\ y'(t) = \delta y(t)\big[-d + \frac{x(t-\tau)}{x(t-\tau)+By(t-\tau)+A}\big], \end{cases} \tag{1.1}$$

where x and y are functions of time representing population densities of prey and predator, respectively; $s, d, \delta > 0$, $A, B \geqslant 0$, $A^2 + B^2 > 0$.

The initial conditions for the system (1.1) take the form of

$$\begin{aligned} &x_0(\theta) = \phi_1(\theta) \geqslant 0, \qquad y_0(\theta) = \phi_2(\theta) \geqslant 0, \\ &\quad \theta \in [-\tau, 0],\ x_0(0) > 0,\ y_0(0) > 0, \end{aligned} \tag{1.2}$$

where $\phi = (\phi_1, \phi_2) \in C([-\tau, 0], \mathbb{R}_+^2)$, $\mathbb{R}_+^2 = \{(x, y)\colon\ x \geqslant 0,\ y \geqslant 0\}$, $\|\phi\| = \max\{|\phi(\theta)|\colon \theta \in [-\tau, 0]\}$, and $|\phi|$ is any norm in $\mathbb{R}^2$.

The functional response in (1.1) was introduced by Beddington [3] and DeAngelis et al. [9]. The system (1.1) satisfies the theorem of existence and uniqueness of solutions under the initial-value condition (1.2).

Cantrell et al. [5] and Hwang [14] considered the following predator–prey system without delay

$$\begin{cases} x'(t) = x(t)(1 - x(t)) - \frac{sx(t)y(t)}{x(t)+By(t)+A}, \\ y'(t) = \delta y(t)\big[-d + \frac{x(t)}{x(t)+By(t)+A}\big], \\ x(0) = x_0 > 0, \qquad y(0) = y_0 > 0. \end{cases} \tag{1.3}$$

They studied the permanence, global stability, and so on, of system (1.3). For a thorough biological background of the model (1.3), see [1,3,8,9].

If $A > 0$ and $B = 0$, system (1.3) reduces to a Kolmogorov type predator–prey system with Michaelis–Menten functional response (see [12,16]):

$$\begin{cases} x'(t) = x(t)(1 - x(t)) - \frac{sx(t)y(t)}{x(t)+A}, \\ y'(t) = \delta y(t)\big[-d + \frac{x(t)}{x(t)+A}\big], \\ x(0) = x_0 > 0, \qquad y(0) = y_0 > 0, \end{cases} \tag{1.4}$$

while if $A = 0$ and $B > 0$, system (1.3) reduces to the ratio-dependent predator–prey system (see [2])

$$\begin{cases} x'(t) = x(t)(1 - x(t)) - \frac{sx(t)y(t)}{x(t)+By(t)}, \\ y'(t) = \delta y(t)\big[-d + \frac{x(t)}{x(t)+By(t)}\big], \\ x(0) = x_0 > 0, \qquad y(0) = y_0 > 0. \end{cases} \tag{1.5}$$

The models (1.4) and (1.5) have been studied by many authors, for details see [15] and its references.

In [4] Beretta and Kuang studied a ratio-dependent predator–prey system with delay τ:

$$\begin{cases} x'(t) = x(t)(1 - x(t)) - \frac{sx(t)y(t)}{x(t)+By(t)}, \\ y'(t) = \delta y(t)\big[-d + \frac{x(t-\tau)}{x(t-\tau)+By(t-\tau)}\big]. \end{cases} \tag{1.6}$$

Clearly, system (1.6) is a special case of system (1.1) with $A = 0$. We also mention that when we almost finish this paper, we know the paper [17] published recently. In [17], the authors studied the stability and Hopf bifurcation of the ratio-dependent predator–prey system (1.6). For Hopf bifurcation in [17], the direction and stability of the Hopf bifurcation were determined, but only at a special and simple situation, i.e., $\alpha_1 = 0$. It should be pointed out that our method in this paper can be easily used to study the general case, i.e., $\alpha_1 = 0$ and $\alpha_1 \neq 0$ together. Furthermore, the ratio-dependent predator–prey system has somewhat singular behavior at low densities and has been criticized on other grounds. For a mathematical analysis and some aspects of the debate among biologists about the ratio dependence, please see [15] and the references in [8]. It is known that the Beddington–DeAngelis form of functional response has some of the same qualitative features as the ratio dependent form but avoids some of the behaviors of the ratio dependent models at low densities which have been the source of controversy (see [5]). In this paper, we will study the predator–prey system (1.1) with Beddington–DeAngelis functional response. From the point of view of biology, we should consider system (1.1) in the closed first quadrant in the (x, y) plane.

This paper is organized as follows. In Section 2, we consider the stability of the interior equilibrium and show that when the delay takes some critical values Hopf bifurcation occurs by choosing delay τ as a bifurcation parameter. In Section 3, following the procedure of deriving normal form due to Faria and Magalhães [10,11], we compute the normal form for the Hopf bifurcation of system (1.1) and study the direction and stability of the Hopf bifurcation. Finally, we illustrate the procedure with a particular example. Numerical simulations support our results.

2. Local stability analysis and Hopf bifurcation

From [5], we know that if $d \geqslant (1 + A)^{-1}$ the equilibrium $(1, 0)$ is globally asymptotically stable and if $0 < d < (1 + A)^{-1}$ there exist three equilibria $(0, 0)$, $(1, 0)$, (x^*, y^*) in the closed first quadrant, where x^* and y^* are positive and satisfy

$$1 - x^* - \frac{sy^*}{x^* + By^* + A} = 0, \qquad \frac{x^*}{x^* + By^* + A} = d. \tag{2.1}$$

In this section, we will consider the local stability of the equilibrium (x^*, y^*) and Hopf bifurcation of system (1.1).

Let $z_1(t) = x(t) - x^*$, $z_2(t) = y(t) - y^*$. System (1.1) becomes

$$\begin{cases} z_1'(t) = \alpha_1 z_1(t) + \alpha_2 z_2(t) + \sum_{i+j\geqslant 2} \frac{1}{i!j!} f_{ij}^{(1)} z_1^i(t) z_2^j(t), \\ z_2'(t) = \beta_1 z_1(t-\tau) + \beta_2 z_2(t-\tau) \\ \qquad\qquad + \sum_{i+j+l\geqslant 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} z_1^i(t-\tau) z_2^j(t-\tau) z_2^l(t), \end{cases} \tag{2.2}$$

where

$$\alpha_1 = -x^* + \frac{sx^*y^*}{(x^* + By^* + A)^2}, \qquad \alpha_2 = -\frac{sx^*(x^* + A)}{(x^* + By^* + A)^2},$$
$$\beta_1 = \frac{\delta y^*(By^* + A)}{(x^* + By^* + A)^2}, \qquad \beta_2 = \frac{-B\delta x^* y^*}{(x^* + By^* + A)^2},$$

$$f_{ij}^{(1)} = \frac{\partial^{i+j} f^{(1)}}{\partial x^i \partial y^j}\bigg|_{(x^*,y^*)}, \quad f_{ijl}^{(2)} = \frac{\partial^{i+j+l} f^{(2)}}{\partial x^i \partial y^j \partial y_1^l}\bigg|_{(x^*,y^*,y^*)}, \quad i,j,l \geqslant 0,$$
$$f^{(1)} = x(1-x) - \frac{sxy}{x+By+A}, \qquad f^{(2)} = \delta y_1\left(-d + \frac{x}{x+By+A}\right).$$

To study the stability of the equilibrium (x^*, y^*), it is sufficient to study the stability of the origin for system (2.2). Consider the linearized system of system (2.2) at $(0,0)$,

$$\begin{cases} z_1'(t) = \alpha_1 z_1(t) + \alpha_2 z_2(t), \\ z_2'(t) = \beta_1 z_1(t-\tau) + \beta_2 z_2(t-\tau). \end{cases} \tag{2.3}$$

System (2.3) has the characteristic equation of the form

$$\lambda^2 - \alpha_1\lambda - \beta_2\lambda e^{-\tau\lambda} + (\alpha_1\beta_2 - \alpha_2\beta_1)e^{-\tau\lambda} = 0. \tag{2.4}$$

In the following, we regard the time delay τ as the parameter to consider the local stability of the equilibrium (x^*, y^*) and the Hopf bifurcation of system (1.1).

Lemma 1. *Suppose* $0 < d < (1+A)^{-1}$, $x^*(dB\delta - ds - s) + ds - dB\delta < 0$. *Then at*

$$\tau_k = \frac{1}{\sigma_+}\left(\arccos\frac{-\alpha_2\beta_1\sigma_+^2}{\beta_2^2\sigma_+^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2} + 2k\pi\right), \quad k = 0,1,\ldots, \tag{2.5}$$

Eq. (2.4) *has a simple pair of conjugate purely imaginary roots* $\pm i\sigma_+$, *where*

$$\sigma_+ = \sqrt{\frac{-(\alpha_1^2 - \beta_2^2) + \sqrt{(\alpha_1^2 - \beta_2^2)^2 + 4(\alpha_1\beta_2 - \alpha_2\beta_1)^2}}{2}}.$$

Furthermore, we have the following:

1. *If* $\tau \in [0, \tau_0)$, *all roots of Eq.* (2.4) *have negative real parts.*
2. *If* $\tau = \tau_0$, *Eq.* (2.4) *has a pair of conjugate purely imaginary roots* $\pm i\sigma_+$, *and all other roots have negative real parts.*

Proof. Clearly, $\lambda = 0$ is not a solution of (2.4). Assume that for some $\tau \geqslant 0$, $i\sigma$ with $\sigma > 0$ is a solution of (2.4). Substituting $\lambda = i\sigma$ into Eq. (2.4) and separating the real and imaginary parts yield

$$\begin{aligned} &-\sigma^2 - \beta_2\sigma\sin\sigma\tau + (\alpha_1\beta_2 - \alpha_2\beta_1)\cos\sigma\tau = 0, \\ &-\sigma\alpha_1 - \beta_2\sigma\cos\sigma\tau - (\alpha_1\beta_2 - \alpha_2\beta_1)\sin\sigma\tau = 0. \end{aligned} \tag{2.6}$$

From (2.6), we have

$$\begin{aligned} &\left(-\sigma^2\right)^2 = \left[\beta_2\sigma\sin\sigma\tau - (\alpha_1\beta_2 - \alpha_2\beta_1)\cos\sigma\tau\right]^2, \\ &(-\sigma\alpha_1)^2 = \left[\beta_2\sigma\cos\sigma\tau + (\alpha_1\beta_2 - \alpha_2\beta_1)\sin\sigma\tau\right]^2, \end{aligned}$$

which imply

$$\sigma^4 + \sigma^2\alpha_1^2 - \beta_2^2\sigma^2 - (\alpha_1\beta_2 - \alpha_2\beta_1)^2 = 0. \tag{2.7}$$

It is easy to see that (2.7) has only one positive real root

$$\sigma_+ = \sqrt{\frac{-(\alpha_1^2 - \beta_2^2) + \sqrt{(\alpha_1^2 - \beta_2^2)^2 + 4(\alpha_1\beta_2 - \alpha_2\beta_1)^2}}{2}}.$$

From (2.6) we know that Eq. (2.4) with $\tau = \tau_k$ $(k = 0, 1, \ldots)$ has a pair of imaginary roots $\pm i\sigma_+$.

Consider the Eq. (2.4) with $\tau = 0$, that is,

$$\lambda^2 - \alpha_1\lambda - \beta_2\lambda + (\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \tag{2.8}$$

From (2.8), we can obtain

$$\lambda = \frac{\alpha_1 + \beta_2 \pm \sqrt{(\alpha_1 + \beta_2)^2 - 4(\alpha_1\beta_2 - \alpha_2\beta_1)}}{2}.$$

It is easy to know that $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$. Clearly, when $\alpha_1 + \beta_2 < 0$, all roots of Eq. (2.8) have negative real parts. From (2.2) and (2.1), we obtain

$$\alpha_1 + \beta_2 = -x^* + \frac{(s - B\delta)x^*y^*}{(x^* + By^* + A)^2} = \frac{x^*(dB\delta - ds - s) + ds - dB\delta}{s}.$$

Thus, all roots of Eq. (2.8) have negative real parts when $x^*(dB\delta - ds - s) + ds - dB\delta < 0$. Applying Rouché theorem (in the form of the lemma in Cooke and Grossman [7]), we obtain the conclusion (1) and (2). This completes the proof. □

Lemma 2. *Denote $\lambda_k(\tau) = \mu_k(\tau) \pm i\sigma_k(\tau)$ as the root of Eq.* (2.4) *satisfying $\mu_k(\tau_k) = 0$, $\sigma_k(\tau_k) = \sigma_+$, $k = 0, 1, \ldots$. For convenience, we omit the subscript k from λ_k, μ_k and σ_k. The following transversality condition*:

$$\operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)\right\}_{\lambda = i\sigma_+} > 0$$

is satisfied.

Proof. By computing, we can know

$$\begin{aligned}
\operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)\right\}_{\lambda = i\sigma_+} &= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\tau}{d\lambda}\right)\right\}_{\lambda = i\sigma_+} \\
&= \operatorname{sign}\left\{\operatorname{Re}\left[\frac{\alpha_1 - 2\lambda}{\lambda(\lambda^2 - \alpha_1\lambda)} + \frac{-\beta_2}{\lambda(-\beta_2\lambda + \alpha_1\beta_2 - \alpha_2\beta_1)}\right]_{\lambda = i\sigma_+}\right\} \\
&= \operatorname{sign}\left\{\frac{\alpha_1^2 + 2\sigma_+^2}{\sigma_+^4 + \alpha_1^2\sigma_+^2} + \frac{-\beta_2^2}{(\alpha_1\beta_2 - \alpha_2\beta_1)^2 + \beta_2^2\sigma_+^2}\right\} = \operatorname{sign}\left\{\alpha_1^2 - \beta_2^2 + 2\sigma_+^2\right\}.
\end{aligned}$$

Equation (2.7) was used in the last step. By inserting the expression for σ_+^2, we see that the sign is positive. The proof is complete. □

By the lemma in [7] and Lemma 2, we obtain the following lemma.

Lemma 3. *Suppose* $0 < d < (1 + A)^{-1}$, $x^*(dB\delta - ds - s) + ds - dB\delta < 0$. *If* $\tau > \tau_0$, *Eq.* (2.4) *has at least one root with strictly positive real part.*

From Lemmas 1–3, we have the following result on stability of zero solution of (2.2).

Lemma 4. *Suppose* $0 < d < (1 + A)^{-1}$, $x^*(dB\delta - ds - s) + ds - dB\delta < 0$. *For system* (2.2), *we have*

1. *If* $\tau \in [0, \tau_0)$, *then the zero solution of* (2.2) *is asymptotically stable.*
2. *If* $\tau > \tau_0$, *the zero solution of* (2.2) *is unstable.*

Summarizing the above discussion and using the standard Hopf bifurcation theorem for retarded FDEs (see [13]), we have the following theorem.

Theorem 1. *Assume* $0 < d < (1 + A)^{-1}$, $x^*(dB\delta - ds - s) + ds - dB\delta < 0$. *For* $k \in N_0$, *denote* $\lambda(\tau) = \mu(\tau) \pm i\sigma(\tau)$ *as the root of Eq.* (2.4) *satisfying*

$$\mu(\tau_k) = 0, \qquad \sigma(\tau_k) = \sigma_+, \qquad \mu'(\tau_k) = \operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)_{\lambda = i\sigma_+} \neq 0,$$

where τ_k *is given in* (2.5). *The other roots* $\lambda(\neq \pm i\sigma_+)$ *of* (2.4) *satisfy* $\lambda \neq im\sigma_+$ *at* $\tau = \tau_k$, *where* m *is any integer. Hence, Hopf bifurcation occurs for* (2.2) *at* $z = 0$ *and* $\tau = \tau_k$.

3. Direction and stability of the Hopf bifurcation

In Section 2 we obtained the conditions which guarantee that the system (2.2) undergoes the Hopf bifurcation at the critical values τ_k, $k = 0, 1, \ldots$. In this section, we will study the direction and stability of the Hopf bifurcation by using the normal form theory of retarded functional differential equations due to Faria and Magalhães [11].

Normalizing the delay τ in system (2.2) by the time-scaling $t \to t/\tau$, (2.2) is transformed into

$$\begin{cases} z_1'(t) = \tau\big[\alpha_1 z_1(t) + \alpha_2 z_2(t) + \sum_{i+j\geqslant 2} \frac{1}{i!j!} f_{ij}^{(1)} z_1^i(t) z_2^j(t)\big], \\ z_2'(t) = \tau\big[\beta_1 z_1(t-1) + \beta_2 z_2(t-1) \\ \qquad + \sum_{i+j+l\geqslant 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} z_1^i(t-1) z_2^j(t-1) z_2^l(t)\big]. \end{cases} \tag{3.1}$$

Now the characteristic equation of the linear equation of the form

$$\begin{cases} z_1'(t) = \tau[\alpha_1 z_1(t) + \alpha_2 z_2(t)], \\ z_2'(t) = \tau[\beta_1 z_1(t-1) + \beta_2 z_2(t-1)] \end{cases} \tag{3.2}$$

is

$$\lambda^2 - \tau\alpha_1\lambda - \tau\beta_2\lambda e^{-\lambda} + \tau^2(\alpha_1\beta_2 - \alpha_2\beta_1)e^{-\lambda} = 0. \tag{3.3}$$

By the translation $\lambda = \zeta\tau$, (3.3) becomes

$$\zeta^2 - \alpha_1\zeta - \beta_2\zeta e^{-\tau\zeta} + (\alpha_1\beta_2 - \alpha_2\beta_1)e^{-\zeta\tau} = 0. \tag{3.4}$$

Clearly, Eq. (3.4) is the same with Eq. (2.4). From the above discussion about Eq. (2.4), we know that for $k \in N_0$, the characteristic equation (3.4) has two simple complex roots $\zeta(\tau) = \mu(\tau) \pm i\sigma(\tau)$ which satisfy

$$\mu(\tau_k) = 0, \qquad \sigma(\tau_k) = \sigma_+, \qquad \mu'(\tau_k) \neq 0,$$

where τ_k is given in (2.5). Then, the characteristic equation (3.3) has two simple complex roots $\lambda(\tau) = \tau\zeta(\tau) = \tau\mu(\tau) \pm i\tau\sigma(\tau)$ which satisfy

$$\tau_k\mu(\tau_k) = 0, \qquad \tau_k\sigma(\tau_k) = \tau_k\sigma_+,$$
$$\left.\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}\right|_{\tau=\tau_k} = \tau\left.\frac{d\,\mathrm{Re}\,\zeta(\tau)}{d\tau}\right|_{\tau=\tau_k} + \mathrm{Re}\,\zeta(\tau)|_{\tau=\tau_k} \neq 0.$$

We write (3.1) in $C := C([-1,0];\mathbb{R}^2)$ as a FDE

$$z'(t) = L(\tau)(z_t) + F(z_t,\tau), \tag{3.5}$$

where

$$L(\tau)(\varphi) = \tau\begin{pmatrix} \alpha_1\varphi_1(0) + \alpha_2\varphi_2(0) \\ \beta_1\varphi_1(-1) + \beta_2\varphi_2(-1)\end{pmatrix},$$
$$F(\varphi,\tau) = \tau\begin{pmatrix} \sum_{i+j\geqslant 2}\frac{1}{i!j!}f_{ij}^{(1)}\varphi_1^i(0)\varphi_2^j(0) \\ \sum_{i+j+l\geqslant 2}\frac{1}{i!j!l!}f_{ijl}^{(2)}\varphi_1^i(-1)\varphi_2^j(-1)\varphi_2^l(0)\end{pmatrix},$$

here $\varphi = \mathrm{col}(\varphi_1,\varphi_2)$. We expand F about φ as the Taylor expansion

$$F(\varphi,\tau) = \frac{1}{2}F_2(\varphi,\tau) + \frac{1}{3!}F_3(\varphi,\tau) + O\big(|\varphi|^4\big), \tag{3.6}$$

where

$$\frac{1}{p!}F_p(\varphi,\tau) = \tau\begin{pmatrix} \sum_{i+j=p}\frac{1}{i!j!}f_{ij}^{(1)}\varphi_1^i(0)\varphi_2^j(0) \\ \sum_{i+j+l=p}\frac{1}{i!j!l!}f_{ijl}^{(2)}\varphi_1^i(-1)\varphi_2^j(-1)\varphi_2^l(0)\end{pmatrix}.$$

Let $k \in N_0$ be fixed. Setting the new parameter $\alpha = \tau - \tau_k$,(3.5) can be rewritten as

$$z'(t) = L(\tau_k)(z_t) + F_0(z_t,\alpha), \tag{3.7}$$

where $F_0(\varphi,\alpha) = L(\alpha)\varphi + F(\varphi,\tau_k+\alpha)$.

Assuming A_0 is the infinitesimal generator of $z'(t) = L(\tau_k)(z_t)$, then A_0 has a pair of conjugate purely imaginary roots $\pm i\sigma_k$, $\sigma_k = \tau_k\sigma_+$. Set $\Lambda = \{-i\sigma_k, i\sigma_k\}$ and denote by P the invariant space of A_0 associated with Λ, where $\dim P = 2$. We can decompose $C := C([-1,0];\mathbb{R}^2)$ as $C = P \oplus Q$ by using the formal adjoint theory for FDEs in [13]. Considering complex coordinates, we still denote $C([-1,0];\mathbb{C}^2)$ as C. Let $\Phi = (\Phi_1,\Phi_2)$ be the bases for P, where

$$\Phi_1(\theta) = e^{i\sigma_k\theta}v, \quad \Phi_2(\theta) = \overline{\Phi_1(\theta)}, \quad -1 \leqslant \theta \leqslant 0,$$

here $v = \binom{v_1}{v_2}$ is a vector in $\mathbb{C}^2$ that satisfies

$$L(\tau_k)(\Phi_1) = i\sigma_k v. \tag{3.8}$$

Choose a basis Ψ for the adjoint space P^*,

$$\Psi(s) = \operatorname{col}\big(\Psi_1(s), \Psi_2(s)\big),$$

where $\Psi_1(s) = e^{-i\sigma_k s}u^T$, $\Psi_2(s) = \overline{\Psi_1(s)}$, $0 \leqslant s \leqslant 1$, here $u = \binom{u_1}{u_2} \in \mathbb{C}^2$. Let $(\Psi, \Phi) = ((\Psi_j, \Phi_i),\ i, j = 1, 2)$, where $(\cdot,\cdot)$ is the bilinear form

$$(\psi, \varphi) := \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta)\, d\eta(\theta)\varphi(\xi)\, d\xi, \quad \forall \varphi \in P,\ \psi \in P^*.$$

From [13], we know that (Ψ, Φ) can be normalized so that $(\Psi, \Phi) = I_2$. By computing, we can choose

$$v = \begin{pmatrix} 1 \\ \frac{i\sigma_k - \tau_k\alpha_1}{\tau_k\alpha_2} \end{pmatrix}, \qquad u = u_1 \begin{pmatrix} 1 \\ \frac{(i\sigma_k - \tau_k\alpha_1)e^{i\sigma_k}}{\tau_k\beta_1} \end{pmatrix}, \tag{3.9}$$

where $u_1 = \frac{\tau_k\beta_1\alpha_2}{2\tau_k\beta_1\alpha_2 + (i\sigma_k - \tau_k\alpha_1)[\beta_2(1 + i\sigma_k - \tau_k\alpha_1) - \alpha_1 e^{i\sigma_k} + \tau_k\alpha_2\beta_1]}$, such that $(\Psi, \Phi) = I_2$. It is known that $\dot{\Phi} = \Phi B$, where B is the 2×2 diagonal matrix

$$B = \begin{pmatrix} i\sigma_k & 0 \\ 0 & -i\sigma_k \end{pmatrix}.$$

Take the enlarged phase space $BC := \{\varphi : [-1, 0] \to \mathbb{C}^2 \mid \varphi$ is continuous on $[-1, 0)$, $\exists \lim_{\theta\to 0^-}\varphi(\theta)\}$. The projection $\varphi \mapsto \Phi(\Psi, \varphi)$ of C upon P, associated with the decomposition $C = P \oplus Q$, is now replaced by $\pi : BC \to P$ such that $\pi(\varphi + X_0\alpha) = \Phi[(\Psi, \varphi) + \Psi(0)\alpha]$. Thus we have the decomposition

$$BC = P \oplus \operatorname{Ker}\pi.$$

Using the decomposition $z_t = \Phi x(t) + y$, $x(t) \in \mathbb{C}^2$, $y \in \operatorname{Ker}\pi \cap C^1 = Q^1$, we decompose (3.7) as

$$\begin{cases} \dot{x} = Bx + \Psi(0)F_0(\Phi x + y, \alpha), \\ \frac{d}{dt}y = A_{Q^1}y + (I - \pi)X_0F_0(\Phi x + y, \alpha), \end{cases} \tag{3.10}$$

where $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & -1 \leqslant \theta < 0. \end{cases}$$

We write the Taylor expansion

$$\Psi(0)F_0(\Phi x + y, \alpha) = \frac{1}{2}f_2^1(x, y, \alpha) + \frac{1}{3!}f_3^1(x, y, \alpha) + \text{h.o.t.},$$
$$(I - \pi)X_0F_0(\Phi x + y, \alpha) = \frac{1}{2}f_2^2(x, y, \alpha) + \frac{1}{3!}f_3^2(x, y, \alpha) + \text{h.o.t.},$$

where $f_j^1(x, y, \alpha)$ and $f_j^2(x, y, \alpha)$ are homogeneous polynomials in (x, y, α) of degree j, $j = 2, 3$, with coefficients in $\mathbb{C}^2$ and $\operatorname{Ker}\pi$, respectively, h.o.t. stands for higher order terms. The normal form method implies a normal form on the center manifold of the origin for (3.7) as

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0, \alpha) + \frac{1}{3!}g_3^1(x, 0, \alpha) + \text{h.o.t.}, \tag{3.11}$$

where g_2^1 and g_3^1 are the second and third order terms in (x,α), respectively, and h.o.t stands for higher order terms. In the following, $V_j^{m+p}(X)$ denotes the linear space of homogeneous polynomials of degree j in $m+p$ real variables, $x=(x_1,\dots,x_m)$, $\alpha=(\alpha_1,\dots,\alpha_p)$ with coefficients in X, and $(M_j^1 p)(x,\alpha)=[B,\,p(\cdot,\alpha)](x)$, where $[B,\,p(\cdot,\alpha)]$ denotes the Lie bracket $[B,\,p(\cdot,\alpha)](x)=D_x p(x,\alpha)Bx-Bp(x,\alpha)$. In this case, the operator M_j^1 acts in $V_j^3(\mathbb{C}^2)$. It is easy to know that $V_j^3(\mathbb{C}^2)=\operatorname{Im}(M_j^1)\oplus\operatorname{Ker}(M_j^1)$, and

$$\operatorname{Ker}\bigl(M_j^1\bigr)=\operatorname{span}\bigl\{x^q\alpha^l e_k\colon\ \bigl(q,\bar\lambda\bigr)=\lambda_k,\ k=1,2,\ q\in N_0^2,\ l\in N_0,\ \bigl|(q,l)\bigr|=j\bigr\},$$

where $\{e_1,e_2\}$ is the canonical basis for $\mathbb{C}^2$ (see [11]). Hence, we have

$$\operatorname{Ker}\bigl(M_2^1\bigr)=\operatorname{span}\left\{\begin{pmatrix}x_1\alpha\\0\end{pmatrix},\begin{pmatrix}0\\x_2\alpha\end{pmatrix}\right\},$$
$$\operatorname{Ker}\bigl(M_3^1\bigr)=\operatorname{span}\left\{\begin{pmatrix}x_1^2x_2\\0\end{pmatrix},\begin{pmatrix}x_1\alpha^2\\0\end{pmatrix},\begin{pmatrix}0\\x_1x_2^2\end{pmatrix},\begin{pmatrix}0\\x_2\alpha^2\end{pmatrix}\right\}.$$

For (3.10), it follows

$$f_2^1(x,y,\alpha)=\Psi(0)\bigl[2L(\alpha)(\Phi x+y)+F_2(\Phi x+y,\tau_k)\bigr], \tag{3.12}$$

where F_2 is given in (3.6). We can obtain

$$f_2^1(x,0,\alpha)=\begin{pmatrix}2A_1x_1\alpha+2A_2\alpha x_2+a_{20}x_1^2+2a_{11}x_1x_2+a_{02}x_2^2\\ 2\bar A_1x_2\alpha+2\bar A_2\alpha x_1+\bar a_{02}x_1^2+2\bar a_{11}x_1x_2+\bar a_{20}x_2^2\end{pmatrix}, \tag{3.13}$$

where

$$\begin{aligned}
A_1&=\frac{i\sigma_k}{\tau_k}u^Tv,\\
A_2&=\frac{-i\sigma_k}{\tau_k}u^T\bar v,\\
a_{20}&=\tau_k\bigl[u_1v_1^2f_{20}^{(1)}+2u_1v_1v_2f_{11}^{(1)}+u_1v_2^2f_{02}^{(1)}+u_2v_1^2f_{200}^{(2)}e^{-2i\sigma_k}+u_2v_2^2f_{020}^{(2)}e^{-2i\sigma_k}\\
&\quad+2u_2v_1v_2f_{110}^{(2)}e^{-i2\sigma_k}+2u_2v_1v_2f_{101}^{(2)}e^{-i\sigma_k}+2u_2v_2^2f_{011}^{(2)}e^{-i\sigma_k}\bigr],\\
a_{11}&=\tau_k\bigl[u_1v_1\bar v_1f_{20}^{(1)}+u_1(v_1\bar v_2+\bar v_1v_2)f_{11}^{(1)}+u_1\bar v_2v_2f_{02}^{(1)}+u_2v_1\bar v_1f_{200}^{(2)}\\
&\quad+u_2v_2\bar v_2f_{020}^{(2)}+u_2(\bar v_1v_2+v_1\bar v_2)f_{110}^{(2)}+u_2\bigl(e^{-i\sigma_k}v_1\bar v_2+e^{i\sigma_k}\bar v_1v_2\bigr)f_{101}^{(2)}\\
&\quad+u_2\bigl(e^{-i\sigma_k}v_2\bar v_2+e^{i\sigma_k}\bar v_2v_2\bigr)f_{011}^{(2)}\bigr],\\
a_{02}&=\tau_k\bigl[u_1\bar v_1^2f_{20}^{(1)}+2u_1\bar v_1\bar v_2f_{11}^{(1)}+u_1\bar v_2^2f_{02}^{(1)}+u_2\bar v_1^2e^{2i\sigma_k}f_{200}^{(2)}+e^{2i\sigma_k}\bar v_2^2u_2f_{020}^{(2)}\\
&\quad+2e^{2i\sigma_k}\bar v_1\bar v_2u_2f_{110}^{(2)}+2e^{i\sigma_k}\bar v_1\bar v_2u_2f_{101}^{(2)}+2e^{i\sigma_k}\bar v_2^2u_2f_{011}^{(2)}\bigr].
\end{aligned}$$

The second order terms in (α,x) of the normal form on the center manifold are given by

$$\frac12 g_2^1(x,0,\alpha)=\frac12\operatorname{Proj}_{\operatorname{Ker}(M_2^1)}f_2^1(x,0,\alpha).$$

It implies

$$\frac12 g_2^1(x,0,\alpha)=\begin{pmatrix}A_1x_1\alpha\\ \bar A_1x_2\alpha\end{pmatrix}, \tag{3.14}$$

where $A_1 = \frac{i\sigma_k}{\tau_k} u^T v$, here $v = \binom{v_1}{v_2}$, $u = \binom{u_1}{u_2}$.

To compute the cubic terms $g_3^1(x, 0, \alpha)$, we note that

$$g_3^1(x, 0, \alpha) \in \operatorname{Ker}(M_3^1) = \operatorname{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\alpha^2 \end{pmatrix} \right\}.$$

However, the terms $O(|x|\alpha^2)$ are irrelevant to determine the generic Hopf bifurcation. Hence, it is only needed to compute the coefficients of

$$\begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}.$$

It follows that

$$\frac{1}{3!} g_3^1(x, 0, \alpha) = \frac{1}{3!} \operatorname{Proj}_{\operatorname{Ker}(M_3^1)} \bar{f}_3^1(x, 0, \alpha) = \frac{1}{3!} \operatorname{Proj}_s \bar{f}_3^1(x, 0, 0) + O(|x|\alpha^2),$$

where

$$s := \operatorname{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \right\},$$

and

$$\bar{f}_3^1(x, 0, 0) = f_3^1(x, 0, 0) + \frac{3}{2}\left[(D_x f_2^1)U_2^1 - (D_x U_2^1) g_2^1\right]_{(x,0,0)} + \frac{3}{2}\left[(D_y f_2^1)h\right]_{(x,0,0)},$$

here $\bar{f}_3^1(x, 0, 0)$ is the third order terms of the equation which is obtained after computing the second order terms of the normal form.

Now we compute $\frac{1}{3!} g_3^1(x, 0, \alpha)$ step by step.

(a) Firstly, we compute $\operatorname{Proj}_s[(D_x f_2^1)U_2^1]_{(x,0,0)}$. Following [11], we take

$$U_2^1(x, 0) = (M_2^1)^{-1} P_{I,2}^1 f_2^1(x, 0, 0).$$

From (3.13), we know

$$f_2^1(x, 0, 0) = \begin{pmatrix} v a_{20} x_1^2 + 2a_{11} x_1 x_2 + a_{02} x_2^2 \\ \bar{a}_{02} x_1^2 + 2\bar{a}_{11} x_1 x_2 + \bar{a}_{20} x_2^2 \end{pmatrix}.$$

A straightforward calculation will show

$$\begin{pmatrix} \frac{1}{i\sigma_k}\left(a_{20} x_1^2 - 2a_{11} x_1 x_2 - \frac{1}{3} a_{02} x_2^2\right) \\ \frac{1}{i\sigma_k}\left(\frac{1}{3}\bar{a}_{02} x_1^2 + 2\bar{a}_{11} x_1 x_2 - \bar{a}_{20} x_2^2\right) \end{pmatrix}.$$

In the end, we obtain by computing

$$\operatorname{Proj}_s\left[(D_x f_2^1)U_2^1\right]_{(x,0,0)} = \begin{pmatrix} \frac{i2}{\sigma_k}\left(a_{20}a_{11} - 2|a_{11}|^2 - \frac{1}{3}|a_{02}|^2\right) x_1^2 x_2 \\ \frac{-i2}{\sigma_k}\left(\overline{a_{20}a_{11}} - 2|a_{11}|^2 - \frac{1}{3}|a_{02}|^2\right) x_1 x_2^2 \end{pmatrix}.$$

(b) Secondly, we compute $[(D_x U_2^1) g_2^1](x, 0, 0)$. From (3.14), we know $g_2^1(x, 0, 0) = 0$. It follows $[(D_x U_2^1) g_2^1](x, 0, 0) = 0$.

(c) Next, we compute $\mathrm{Proj}_s[(D_y f_2^1)h]_{(x,0,0)}$, here h is a second order homogeneous polynomial in (x_1, x_2, α) with coefficients in Q^1. Let

$$h = h(x_1, x_2, \alpha) = h_{110}x_1x_2 + h_{101}x_1\alpha + h_{011}x_2\alpha + h_{200}x_1^2 + h_{020}x_2^2 + h_{002}\alpha^2,$$

where $h = \mathrm{col}(h^1, h^2)$. Following [11], we know that $h = h(x_1, x_2, \alpha)$ is the unique solution in $V_2^3(Q^1)$ of

$$\big(M_2^2 h\big)(x, \alpha) = (I - \pi)X_0\big[2L(\alpha)(\Phi x) + F_2(\Phi x, \tau_k)\big]. \tag{3.15}$$

Since

$$\begin{aligned}\big(M_2^2 h\big)(x, \alpha) &= D_x h(x, \alpha)Bx - A_{Q^1}\big(h(x, \alpha)\big)\\ &= D_x h(x, \alpha)Bx - \dot{h}(x, \alpha) - X_0\big[L(\tau_k)\big(h(x, \alpha) - \dot{h}(x, \alpha)\big)(0)\big]\\ &= (I - \pi)X_0\big[2L(\alpha)(\Phi x) + F_2(\Phi x, \tau_k)\big],\end{aligned}$$

it follows that $h = h(x, 0)(\theta)$ can be evaluated by the system

$$\dot{h}(x) - D_x h(x)Bx = \Phi\Psi(0)\big[2L(0)(\Phi x) + F_2(\Phi x, \tau_k)\big], \tag{3.16}$$

$$\dot{h}(x)(0) - L(\tau_k)\big(h(x)\big) = 2L(0)(\Phi x) + F_2(\Phi x, \tau_k), \tag{3.17}$$

where $\dot{h}$ denotes the derivative of $h(x)\theta$ with respect to θ. From (3.12), we know

$$f_2^1(x, y, 0) = \Psi(0)\big[F_2(\Phi x + y, \tau_k)\big].$$

It follows that

$$\begin{aligned}&f_2^1(x, y, 0)\\ &\quad= \begin{pmatrix} u_1 & u_2 \\ \bar{u}_1 & \bar{u}_2 \end{pmatrix}\left[\tau_k\begin{pmatrix} f_{20}^{(1)}p_1^2 + 2f_{11}^{(1)}p_1p_2 + f_{02}^{(1)}p_2^2 \\ f_{200}^{(2)}l_1^2 + f_{020}^{(2)}l_2^2 + 2f_{110}^{(2)}l_1l_2 + 2f_{101}^{(2)}l_1p_2 + 2f_{011}^{(2)}l_2p_2 \end{pmatrix}\right],\end{aligned}$$

where

$$\begin{aligned}p_1 &= v_1x_1 + \bar{v}_1x_2 + y_1(0),\\ p_2 &= v_2x_1 + \bar{v}_2x_2 + y_2(0),\\ l_1 &= e^{-i\sigma_k}v_1x_1 + e^{i\sigma_k}\bar{v}_1x_2 + y_1(-1),\\ l_2 &= e^{-i\sigma_k}v_2x_1 + e^{i\sigma_k}\bar{v}_2x_2 + y_2(-1).\end{aligned}$$

Then, we can obtain

$$\begin{aligned}&\big[(D_y f_2^1)h\big]_{(x,0,0)}\\ &\quad= \begin{pmatrix} u^T\tau_k\begin{pmatrix} 2f_{20}^{(1)}p_1'h^1(0) + 2f_{11}^{(1)}p_1'h^2(0) + 2f_{11}^{(1)}p_2'h^1(0) + 2f_{02}^{(1)}p_2'h^2(0) \\ Q_1 + Q_2 \end{pmatrix} \\ \bar{u}^T\tau_k\begin{pmatrix} 2f_{20}^{(1)}p_1'h^1(0) + 2f_{11}^{(1)}p_1'h^2(0) + 2f_{11}^{(1)}p_2'h^1(0) + 2f_{02}^{(1)}p_2'h^2(0) \\ Q_1 + Q_2 \end{pmatrix} \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}
&p_1' = v_1 x_1 + \bar{v}_1 x_2,\\
&p_2' = v_2 x_1 + \bar{v}_2 x_2,\\
&Q_1 = 2 f_{200}^{(2)} l_1' h^1(-1) + 2 f_{020}^{(2)} l_2' h^2(-1) + 2 f_{110}^{(2)}\big(l_1' h^2(-1) + l_2' h^1(-1)\big),\\
&Q_2 = 2 f_{101}^{(2)}\big(l_1' h^2(0) + p_2' h^1(-1)\big) + 2 f_{011}^{(2)}\big(l_2' h^2(0) + p_2' h^2(-1)\big),\\
&l_1' = e^{-i\sigma_k} v_1 x_1 + e^{i\sigma_k} \bar{v}_1 x_2,\\
&l_2' = e^{-i\sigma_k} v_2 x_1 + e^{i\sigma_k} \bar{v}_2 x_2.
\end{aligned} \tag{3.18}$$

Hence,

$$\mathrm{Proj}_s\big[\big(D_y f_2^1\big)h\big]_{(x,0,0)} = \begin{pmatrix} 2c_3 x_1^2 x_2 \\ 2\bar{c}_3 x_1 x_2^2 \end{pmatrix},$$

where

$$c_3 = u^T \tau_k \begin{pmatrix} A h_{110}^1(0) + \bar{A} h_{200}^1(0) + B h_{110}^2(0) + \bar{B} h_{200}^2(0) \\ C h_{110}^1(-1) + \bar{C} h_{200}^1(-1) + D h_{110}^2(-1) + \bar{D} h_{200}^2(-1) + E h_{110}^2(0) + \bar{E} h_{200}^2(0) \end{pmatrix}, \tag{3.19}$$

here

$$\begin{aligned}
&A = f_{20}^{(1)} v_1 + f_{11}^{(1)} v_2,\\
&B = f_{11}^{(1)} v_1 + f_{02}^{(1)} v_2,\\
&C = f_{200}^{(2)} v_1 e^{-i\sigma_k} + f_{110}^{(2)} v_2 e^{-i\sigma_k} + f_{101}^{(2)} v_2,\\
&D = f_{020}^{(2)} v_2 e^{-i\sigma_k} + f_{110}^{(2)} v_1 e^{-i\sigma_k} + f_{011}^{(2)} v_2,\\
&E = f_{101}^{(2)} v_1 e^{-i\sigma_k} + f_{011}^{(2)} v_2 e^{-i\sigma_k}.
\end{aligned}$$

We still need to compute $h_{110}(\theta)$ and $h_{200}(\theta)$. From (3.16) and (3.17), we know that $h_{110} = \mathrm{col}(h_{110}^1, h_{110}^2)$ is the solution of

$$\dot{h}_{110} = (\Phi_1 \Phi_2) \begin{pmatrix} 2a_{11} \\ 2\bar{a}_{11} \end{pmatrix}$$

such that

$$\dot{h}_{110}(0) - L(\tau_k)(h_{110}) = \tau_k \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

and $h_{200} = \mathrm{col}(h_{200}^1, h_{200}^2)$ is the solution of

$$\dot{h}_{200} - 2i\sigma_k h_{200} = (\Phi_1 \Phi_2) \begin{pmatrix} a_{20} \\ \bar{a}_{02} \end{pmatrix}$$

such that

$$\dot{h}_{200}(0) - L(\tau_k)(h_{200}) = \tau_k \begin{pmatrix} a_2 \\ b_2 \end{pmatrix},$$

where

$$
\begin{aligned}
a_1 &= 2v_1\bar{v}_1 f_{20}^{(1)} + 2(v_1\bar{v}_2 + \bar{v}_1 v_2) f_{11}^{(1)} + 2\bar{v}_2 v_2 f_{02}^{(1)},\\
b_1 &= 2v_1\bar{v}_1 f_{200}^{(2)} + 2v_2\bar{v}_2 f_{020}^{(2)} + 2(\bar{v}_1 v_2 + v_1\bar{v}_2) f_{110}^{(2)} + 2\big(e^{-i\sigma_k} v_1\bar{v}_2 + e^{i\sigma_k}\bar{v}_1 v_2\big) f_{101}^{(2)}\\
&\quad + 2\big(e^{-i\sigma_k} v_2\bar{v}_2 + e^{i\sigma_k}\bar{v}_2 v_2\big) f_{011}^{(2)},\\
a_2 &= v_1^2 f_{20}^{(1)} + 2v_1 v_2 f_{11}^{(1)} + v_2^2 f_{02}^{(1)},\\
b_2 &= v_1^2 f_{200}^{(2)} e^{-2i\sigma_k} + v_2^2 f_{020}^{(2)} e^{-2i\sigma_k} + 2v_1 v_2 f_{110}^{(2)} e^{-i2\sigma_k} + 2v_1 v_2 f_{101}^{(2)} e^{-i\sigma_k}\\
&\quad + 2v_2^2 f_{011}^{(2)} e^{-i\sigma_k}.
\end{aligned}
$$

Solving for $h_{110}(\theta)$ and $h_{200}(\theta)$, we obtain

$$
h_{110}(\theta) = \frac{2}{i\sigma_k}\big(a_{11}e^{i\sigma_k\theta}v - \bar{a}_{11}e^{-i\sigma_k\theta}\bar{v}\big) + c_1, \tag{3.20}
$$

$$
h_{200}(\theta) = -\frac{1}{i\sigma_k}\left(a_{20}e^{i\sigma_k\theta}v + \frac{1}{3}\bar{a}_{02}e^{-i\sigma_k\theta}\bar{v}\right) + e^{2i\sigma_k\theta}c_2, \tag{3.21}
$$

where $c_1 = (c_1^{(1)}, c_1^{(2)})^T$, $c_2 = (c_2^{(1)}, c_2^{(2)})^T$,

$$
\begin{aligned}
c_1^{(1)} &= \frac{a_1\beta_2 - b_1\alpha_2}{\beta_1\alpha_2 - \alpha_1\beta_2},\\
c_1^{(2)} &= \frac{a_1\beta_1 - b_1\alpha_1}{\beta_2\alpha_1 - \alpha_2\beta_1},\\
c_2^{(1)} &= \frac{\tau_k a_2(2i\sigma_k - \tau_k\beta_2 e^{-2i\sigma_k}) + \tau_k^2 b_2\alpha_2}{(2i\sigma_k - \tau_k\alpha_1)(2i\sigma_k - \tau_k\beta_2 e^{-2i\sigma_k}) - \tau_k^2\alpha_2\beta_1 e^{-2i\sigma_k}},\\
c_2^{(2)} &= \frac{\tau_k^2 a_2\beta_1 e^{-2i\sigma_k} + \tau_k b_2(2i\sigma_k - \tau_k\alpha_1)}{(2i\sigma_k - \tau_k\alpha_1)(2i\sigma_k - \tau_k\beta_2 e^{-2i\sigma_k}) - \tau_k^2\alpha_2\beta_1 e^{-2i\sigma_k}}.
\end{aligned}
$$

(d) Finally, we compute $\mathrm{Proj}_s\, f_3^1(x,0,0)$.
$f_3^1(x,0,0)$ is given by

$$
f_3^1(x,0,0) = \Psi(0)\big[F_3(\Phi x, \tau_k)\big],
$$

where F_3 is defined in (3.6). By computing, F_3 can be represented in

$$
\begin{aligned}
&F_3(\Phi x, \tau_k)\\
&\quad = \tau_k\begin{pmatrix} f_{30}^{(1)}p_1'^3 + 3f_{21}^{(1)}p_1'^2p_2' + 3f_{12}^{(1)}p_1'p_2'^2 + f_{03}^{(1)}p_2'^3 \\ f_{300}^{(2)}l_1'^3 + 3f_{210}^{(2)}l_1'^2l_2' + 3f_{201}^{(2)}l_1'^2p_2' + 3f_{021}^{(2)}l_2'^2p_2' + 3f_{120}^{(2)}l_1'l_2'^2 + f_{030}^{(2)}l_2'^3 + 6f_{111}^{(2)}l_1'l_2'p_2' \end{pmatrix},
\end{aligned}
$$

where p_1', p_2', l_1', l_2' are defined in (3.18). Thus, we obtain

$$
\mathrm{Proj}_s\, f_3^1(x,0,0) = \begin{pmatrix} 3a_{21}x_1^2x_2 \\ 3\bar{a_{21}}x_1x_2^2 \end{pmatrix},
$$

where

$$\begin{aligned}
a_{21} = {} & \tau_k u_1 \big[f_{30}^{(1)} v_1^2 \bar{v}_1 + f_{21}^{(1)} \big(v_1^2 \bar{v}_2 + 2 v_1 \bar{v}_1 v_2 \big) + f_{12}^{(1)} \big(v_2^2 \bar{v}_1 + 2 v_2 \bar{v}_2 v_1 \big) + f_{03}^{(1)} v_2^2 \bar{v}_2 \big] \\
& + \tau_k u_2 \big[f_{300}^{(2)} e^{-i\sigma_k} v_1^2 \bar{v}_1 + f_{210}^{(2)} \big(e^{-i\sigma_k} v_1^2 \bar{v}_2 + 2 e^{-i\sigma_k} v_1 \bar{v}_1 v_2 \big) \\
& + f_{201}^{(2)} \big(e^{-2i\sigma_k} v_1^2 \bar{v}_2 + 2 v_1 \bar{v}_1 v_2 \big) + f_{021}^{(2)} \big(e^{-2i\sigma_k} v_2^2 \bar{v}_2 + 2 \bar{v}_2 v_2^2 \big) \\
& + f_{120}^{(2)} \big(e^{-i\sigma_k} v_2^2 \bar{v}_1 + 2 e^{-i\sigma_k} v_1 \bar{v}_2 v_2 \big) + f_{030}^{(2)} e^{-i\sigma_k} v_2^2 \bar{v}_2 \\
& + 2 f_{111}^{(2)} \big(e^{-2i\sigma_k} v_1 v_2 \bar{v}_2 + v_1 v_2 \bar{v}_2 + v_2^2 \bar{v}_1 \big) \big]. \qquad (3.22)
\end{aligned}$$

Summarizing (a)–(d), we obtain

$$\frac{1}{3!} g_3^1(x, 0, 0) = \begin{pmatrix} A_3 x_1^2 x_2 \\ \bar{A}_3 x_1 x_2^2 \end{pmatrix},$$

where

$$A_3 = \frac{i}{2\sigma_k} \left(a_{20} a_{11} - 2|a_{11}|^2 - \frac{1}{3} |a_{02}|^2 \right) + \frac{1}{2} (c_3 + a_{21}). \qquad (3.23)$$

Thus, the normal form (3.11) has the form

$$\begin{aligned}
\dot{x} &= Bx + \frac{1}{2} g_2^1(x, 0, \alpha) + \frac{1}{3!} g_3^1(x, o, \alpha) + \text{h.o.t.} \\
&= Bx + \begin{pmatrix} A_1 x_1 \alpha \\ \bar{A}_1 x_2 \alpha \end{pmatrix} + \begin{pmatrix} A_3 x_1^2 x_2 \\ \bar{A}_3 x_1 x_2^2 \end{pmatrix} + O\big(|x| \alpha^2 + |x|^4 \big).
\end{aligned}$$

The normal form (3.11) relative to P can be written in real coordinates (w_1, w_2) through the change of variables $x_1 = w_1 - i w_2$, $x_2 = w_1 + i w_2$. Setting $w_1 = \rho \cos \xi$, $w_2 = \rho \sin \xi$, this normal form becomes

$$\begin{cases} \dot{\rho} = k_1 \alpha \rho + k_2 \rho^3 + O(\alpha^2 \rho + |(\rho, \alpha)|^4), \\ \dot{\xi} = -\sigma_k + O(|(\rho, \alpha)|), \end{cases} \qquad (3.24)$$

where $k_1 = \operatorname{Re} \mathrm{A}_1$, $k_2 = \operatorname{Re} \mathrm{A}_3$. Following [6], we know that the sign of $k_1 k_2$ determines the direction of the bifurcation and that the sign of k_2 determines the stability of the nontrivial periodic orbits.

Summarizing, we have the following theorem.

Theorem 2. *The flow of Eq.* (3.7) *on the center manifold of the origin at* $\alpha = 0$ *is given by* (3.24). *Furthermore, we have the following*

(i) *Hopf bifurcation is supercritical if* $k_1 k_2 < 0$, *and subcritical if* $k_1 k_2 > 0$;
(ii) *the nontrivial periodic solution is stable if* $k_2 < 0$, *and unstable if* $k_2 > 0$.

Remark. Even though the normal form procedure for RFDE is given in [11], hard computation is involved. The computation of normal form for RFDE is more difficult than that for ODE.

4. An example and numerical simulations

As an example, consider system (1.1) with $s=1$, $B=1/2$, $d=1/2$, $A=1/9$, $\delta=1$,

$$\begin{cases} x'(t)=x(t)(1-x(t))-\frac{x(t)y(t)}{x(t)+\frac{1}{2}y(t)+\frac{1}{9}}, \\ y'(t)=y(t)\left[-\frac{1}{2}+\frac{x(t-\tau)}{x(t-\tau)+\frac{1}{2}y(t-\tau)+\frac{1}{9}}\right]. \end{cases} \tag{4.1}$$

We get the equilibrium $(x^*,y^*)=(1/3,4/9)$. Furthermore, we also can obtain

$$\sigma_+=\frac{1}{12}\sqrt{2(1+\sqrt{65})},\qquad \tau_k=\frac{1}{\sigma_+}\left(\arccos\frac{36\sigma_+^2}{9\sigma_+^2+4}+2k\pi\right)\quad\text{and}\quad \sigma_k=\tau_k\sigma_+.$$

In the following, we mainly consider the direction and stability of Hopf bifurcation at $\tau=\tau_0=1.3785$. Applying Theorems 1 and 2 to (4.1), we obtain the following theorem.

Theorem 3. *The characteristic equation of* (4.1) *at* $(x^*,y^*)=(1/3,4/9)$ *has two simple roots* $\pm i\sigma_0$ *for* $\tau=\tau_0$ *and the other roots* λ $(\neq\pm i\sigma_0)$ *have non-zero real parts. Furthermore, there is a supercritical Hopf bifurcation for* (4.1) *at* $\tau=\tau_0$ *and the nontrivial periodic solution associated with Hopf bifurcation at* $\tau=\tau_0$ *is stable in the center manifold.*

Proof. From Lemma 1 and Theorem 1, we can obtain that the characteristic equation of (4.1) at $(x^*,y^*)=(1/3,4/9)$ has two simple roots $\pm i\sigma_0$ for $\tau=\tau_0$ and the other roots λ $(\neq\pm i\sigma_0)$ have non-zero real parts. Furthermore, if $\tau\in[0,\tau_0)$, all roots of the characteristic equation of (4.1) at $(x^*,y^*)=(1/3,4/9)$ have negative real parts and the equilibrium $(x^*,y^*)=(1/3,4/9)$ is locally asymptotically stable. According to the above procedure of deriving normal form and Theorem 2, we consider the direction and stability of Hopf bifurcation at $\tau=\tau_0$. In the following, we omit the complicated expressions and directly obtain the numerical results by means of the software Maple.

At first, according to the expressions in (3.9), we can obtain the vectors v and u as follows:

$$v=\begin{pmatrix} v_1\\ v_2\end{pmatrix}=\begin{pmatrix}1\\ -1.0643i\end{pmatrix},\qquad u=\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0.4275-0.2509i\\ 0.0220+0.5271i\end{pmatrix}.$$

From the expressions in (3.13), (3.19)–(3.22), it follows that

$$\begin{aligned} a_{20}&=0.1481+0.9392i,\\ a_{11}&=-0.2773-0.3709i,\\ a_{02}&=-2.0378-0.8972i,\\ a_{21}&=-2.2336+1.9800i,\\ c_3&=-1.5476-1.6707i. \end{aligned}$$

Finally, from (3.14) and (3.23), we have

$$A_1=0.0973+0.3507i,\qquad A_3=-1.5682-1.6592i.$$

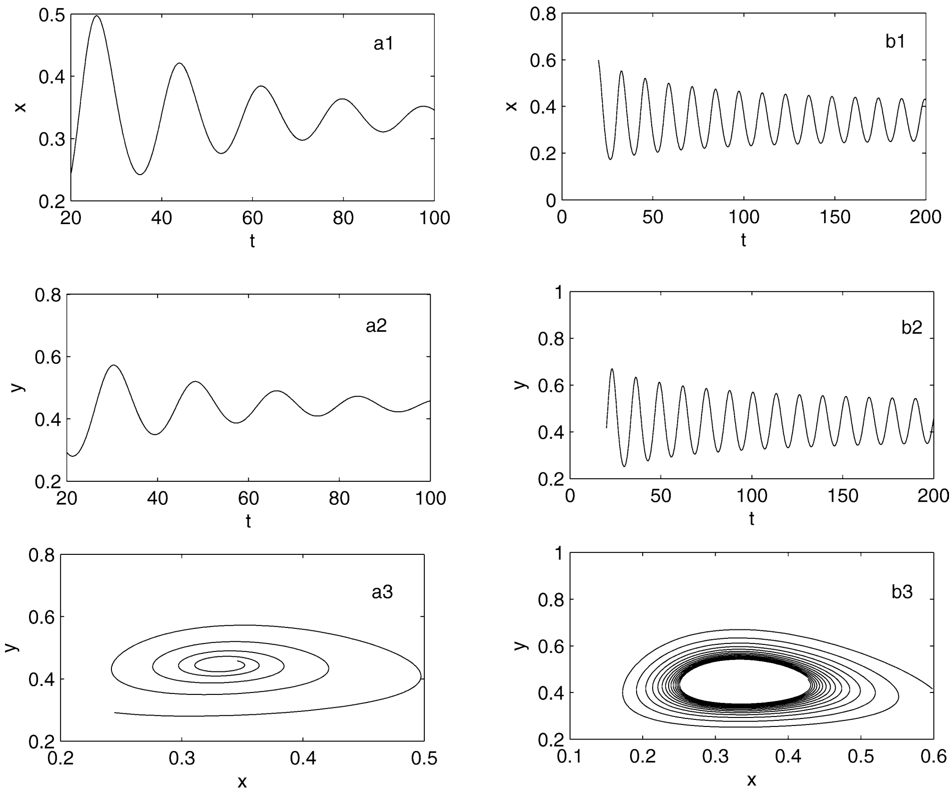

Fig. 1. The time histories and phase trajectories of the system (4.1) before and after Hopf bifurcation occurs. a1–a3: $\tau = 1$; b1–b3: $\tau = 1.4$.

Thus, $k_1 = 0.0973$, $k_2 = -1.5682$. From Theorem 2, we know that there is a supercritical Hopf bifurcation for (4.1) at $\tau = \tau_0$ and the nontrivial periodic solution associated with Hopf bifurcation at $\tau = \tau_0$ is stable in the center manifold. The proof is complete. □

The computer simulations are depicted in Fig. 1 a1–a3 and b1–b3.

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